On Δ-spaces X and their characterization in terms of spaces $C_p(X)$

ABSTRACT

Reed (see [7], [4]) studied those uncountable subsets D (under the name Δ-sets) of the reals $\mathbb{R}$ (with the natural topology) having the following property:

For any decreasing sequence $(H_n)_n$ of subsets of D with $\bigcap_n H_n = \emptyset$ there is a sequence $(V_n)_n$ of G_δ-subsets of D such that $H_n \subset V_n$, $n \in \mathbb{N}$, and $\bigcap_n V_n = \emptyset$. Przymusiński showed [6] that the existence a Δ-set in $\mathbb{R}$ is equivalent to the existence of a countably paracompact separable Moore space not being normal. Research about Δ-spaces is strictly connected with a study of $\mathbb{Q}$-sets, one of the most mysterious objects in $\mathbb{R}$.

In the paper [2] the concept of a Δ-set has been extended to arbitrary topological spaces: A topological space X is called a Δ-*space* if for every decreasing sequence $(D_n)_n$ of subsets of X with $\bigcap_n D_n = \emptyset$, there is a decreasing sequence $(V_n)_n$ of open subsets of X, $D_n \subset V_n$ for every $n \in N$ and $\bigcap_n V_n = \emptyset$.

In [2] we proved that X is a Δ-space if and only if $C_p(X)$ is *distinguished*, i.e. the dual of $C_p(X)$ endowed with the topology of the uniform convergence on $C_p(X)$-bounded sets carries the finest locally convex topology. This analytic approach provided several new results about Δ-sets and Δ-spaces [2], [3] [5]. Some alternative characterization was also presented in [1]. Among the others we proved that every $\check{C}$ech-complete Δ-space is scattered and every scattered Eberlein compact space is a Δ-space. Nevertheless, compact scattered spaces X not being a Δ-space do exist, for example $X = [0, \omega_1]$. Every metrizable scattered space is a Δ-space. Applications for Banach spaces $C(K)$ and spaces $C_p(K)$ are provided.

[1] J. C. Ferrando, S. A. Saxon, *If not distinguished, is $C_p(X)$ even close?*, Proc. Amer. Math. Soc. 149 (2021), 2583-2596.

[2] J. Kąkol, A. Leiderman, *A characterization of X for which spaces $C_p(X)$ are distinguished and its applications*, Proc Amer. Math. Soc. 8 (2021), 86–99.

[3] J. Kąkol, A. Leiderman, *Basic properties of X for which spaces $C_p(X)$ are distinguished*, Proc. Amer. Math. Soc. 8 (2021), 257–280.

[4] R. W. Knight, Δ-*Sets*, Trans. Amer. Math. Soc. 339 (1993), 45-60.

[5] A. Leiderman, V. V. Tkachuk, *Pseudocompact Δ-spaces are often scattered*, Monatshefte für Math. 196 (2021).

[6] T. C. Przymusiński, *Normality and separability of Moore spaces*, Set-Theoretic Topology, Acad. Press, New York, 1977, 325-337.

[7] G. M. Reed, *On normality and countable paracompactness*, Fund. Math. 110 (1980), 145-152.